ISRAEL JOURNAL OF MATHEMATICS, Vol. 24, Nos. 3–4, 1976

GENERAL DECOMPOSITION THEOREMS FOR m-CONVEX SETS IN THE PLANE

BY
MARILYN BREEN AND DAVID C. KAY

ABSTRACT

A set S in R^d is said to be m-convex, $m \geqq 2$, if and only if for every m distinct points in S, at least one of the line segments determined by these points lies in S. Clearly any union of $m-1$ convex sets is m-convex, yet the converse is false and has inspired some interesting mathematical questions: Under what conditions will an m-convex set be decomposable into $m-1$ convex sets? And for every $m \geqq 2$, does there exist a $\sigma(m)$ such that every m-convex set is a union of $\sigma(m)$ convex sets? Pathological examples convince the reader to restrict his attention to closed sets of dimension $\leqq 3$, and this paper provides answers to the questions above for closed subsets of the plane.

If S is a closed m-convex set in the plane, $m \geqq 2$, the first question may be answered in one way by the following result: If there is some line H supporting S at a point p in the kernel of S, then S is a union of $m-1$ convex sets. Using this result, it is possible to prove several decomposition theorems for S under varying conditions. Finally, an answer to the second question is given: If $m \geqq 3$, then S is a union of $(m-1)^3 2^{m-3}$ or fewer convex sets.

1. Introduction

Let S be a subset of R^d. The set S is said to be *m-convex*, $m \geqq 2$, if and only if for every m distinct points in S, at least one of the line segments determined by these points lies in S. A point x in S is said to be a *point of local convexity of S* if and only if there is some neighborhood N of x such that if $y, z \in S \cap N$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of S*. The following familiar terminology will be used: For x, y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. Points $x_1, \cdots, x_n$ in S are *visually independent via S* if and only if for $1 \leqq i < j \leqq n$, x_i does not see x_j via S. Throughout the paper, conv S, ker S, bdry S, and cl S will be used to denote the convex hull of S, the kernel of S, the boundary of S, and the closure of S, respectively. For convenience, Q will represent the set of lnc points of S.

Received March 15, 1975 and in revised form April 2, 1976

Several interesting decomposition theorems have been obtained for closed 3-convex sets in the plane. Valentine [9] has proved that a closed planar 3-convex set S may be written as a union of three or fewer convex sets. If, in addition, S is bounded and has some point of local convexity in bdry $S \cap \ker S$, then by a result of Stamey and Marr [7], S is a union of two convex sets.

For m-convex sets, we have the following analogue (Breen, [1]): For S a closed planar m-convex set with lnc points in Q, if $\operatorname{conv} Q \subseteq S$ and $[(\operatorname{bdry} S) \cap (\ker S)] \sim Q \neq \emptyset$, then S is a union of $m-1$ closed convex sets. However, few other results have been obtained for the general case. Examples by Kay and Guay [5, Example 4] show that such a generalization must require an unpleasantly large number of convex sets, and it was only recently proved by Eggleston [2] that a compact planar m-convex set is expressible as a finite union of convex sets. Here we establish actual bounds for Eggleston's theorem using entirely different methods of proof. Several smaller bounds are obtained in case $\ker S \neq \emptyset$ or $\operatorname{conv} Q \subseteq S$. Also, for $m = 4$, the bound of 6 is established.

2. The case for $\ker S \neq \emptyset$

Theorem 1 employs a basic construction introduced in [1] to generalize a result of that paper. The following theorem by Lawrence, Hare and Kenelly [6, Theorem 2] will be useful throughout the proof.

Lawrence, Hare, Kenelly Theorem. *Let T be a subset of a linear space such that each finite subset $F \subseteq T$ has a k-partition $\{F_1, \cdots, F_k\}$, where* $\operatorname{conv} F_i \subseteq T$, $1 \leqq i \leqq k$. *Then T is a union of k convex sets.*

Theorem 1. *Let S be a closed m-convex set in the plane, $m \geqq 2$. Let $p \in \ker S \neq \emptyset$, and for H some line containing p, assume that $S \subseteq \operatorname{cl} H_1$ (where H_1 is one of the open halfspaces determined by H). Then S is the union of $m-1$ convex sets. The result is best possible for all m.*

Proof. The proof will require several steps: First we show that we may assume S to be bounded and Q to be finite (Lemma 1), with $p \notin Q$ (Lemma 2). Then we consider the collection of rays at p consisting of rays of the form $R(p, q_i)$ for q_i in Q, together with rays R_1 and R_2, where $R_1 \cup R_2 = H$. Order the rays appropriately. Each pair of consecutive rays will define a convex subset of S called a wedge, and we decompose S by defining $m-1$ collections of wedges (Lemmas 3 and 4).

We begin by noticing that we may restrict our attention to the case in which S is bounded: For F any finite subset of S, F lies in some compact disk B, $B \cap S$

satisfies the hypothesis above, and by the Lawrence, Hare, Kenelly Theorem it suffices to prove the result for $B \cap S$. Therefore we shall assume that S is bounded.

LEMMA 1. *To any finite subset* $F = \{x_i : 1 \leqq i \leqq n\}$ *of* S *there corresponds an* m*-convex set* T *having finitely many lnc points, with* $F \subseteq T \subseteq S$. *Hence we may assume that* S *has finitely many lnc points.*

PROOF OF LEMMA 1. Let R_1, R_2 be closed rays at p, with $R_1 \cup R_2 = H$. Consider the family $\mathcal{R}$ of rays consisting of R_1, R_2 together with rays of the form $R(p,q)$ emanating from p through q for some q in Q. It is not hard to show that for R in $\mathcal{R}$, R contains at most two members of Q. Any two (not necessarily distinct) rays in $\mathcal{R}$ bound a closed subset of S, and we let $\mathcal{W}$ denote the collection of all these closed regions. Moreover, since Q is closed, to every point x of S there corresponds a minimal member A of $\mathcal{W}$ which contains x.

Now let $F = \{x_i : 1 \leqq i \leqq n\}$ be a finite subset of S. To each x_i there corresponds a minimal member A_i of $\mathcal{W}$ which contains x_i. Each lnc point of S in A_i must lie in one of the boundary rays of A_i; hence A_i contains at most four members of $Q \sim \{p\}$.

By arguments given in [1], we may assume that no A_i is a segment and also that $A_i = \mathrm{cl}(\mathrm{int}\, A_i)$. (In case $p \in Q$, the A_i sets are not necessarily convex; however, standard arguments show each component of $A_i \sim \{p\}$ to be convex.) Now order the rays associated with the A_i sets in a clockwise direction from R_1 (for an appropriate labeling of R_1, R_2). This in turn induces an order among the A_i sets, and we may relabel the A_i and corresponding x_i so that for $i < j$, the rays defining A_i precede the rays defining A_j in our clockwise ordering. Since $A_i = \mathrm{cl}(\mathrm{int}\, A_i)$, then each wedge is associated with at most two lnc points from $Q \sim \{p\}$, denoted q_i, q'_i. (In case A_i is associated with one lnc point in $Q \sim \{p\}$, then A_i must be bounded by R_1 or R_2, and then we let q_1 be the last point of S on R_1, q'_n the last point of S on R_2.)

By the Lawrence, Hare, Kenelly Theorem, we may assume that each A_i has polygonal boundary, and we may select p_i, p'_i so that $[p_i, q_i]$ and $[p'_i, q'_i]$ lie in $\mathrm{bdry}\, A_i$.

For $1 \leqq i \leqq n-1$, let B_i denote the union of all segments $[x,y]$, where $[x,y] \subseteq S$, $x \in [p, q'_i]$, $y \in [p, q_{i+1}]$. We assert that $(\mathrm{conv}\, B_i) \sim B_i$ is convex and $(\mathrm{bdry\ conv}\, B_i) \sim B_i$ is polygonal: For s, t in $(\mathrm{conv}\, B_i) \sim B_i$, if $[s,t]$ were not in $(\mathrm{conv}\, B_i) \sim B_i$, then (s,t) would contain some point u in B_i, and for some x in $[p, q'_i]$, y in $[p, q_{i+1}]$, $u \in (x,y) \subseteq S$. But then one of s, t would lie on the p side of the line $L(x, y)$, clearly impossible since $\operatorname{conv}\{p, x, y\} \subseteq B_i$. Therefore $[s, t] \subseteq (\operatorname{conv} B_i) \sim B_i$, and the set is convex. Since S is closed, $(\operatorname{bdry} \operatorname{conv} B_i) \sim [q'_i, q_{i+1}] \subseteq S$, and since S is m-convex, it is easy to see that $(\operatorname{bdry} \operatorname{conv} B_i) \sim [q'_i, q_{i+1}]$ is polygonal and consists of at most $m-1$ segments. Hence the set B_i has at most $m-2$ lnc points.

Define $T = \cup\{A_i \cup B_i \cup A_n : 1 \leqq i \leqq n-1\}$. We assert that T is closed and m-convex, that the set Q_T of lnc points of T is finite, and that $p \in \ker T$. For any m-point subset of T, at least one of the corresponding segments, say $[v, w]$, is in S. We will show that $[v, w] \subseteq T$. In case $v, w \in A_i$ for some i, the result is trivial. If $v, w \in B_i$, then if (v, w) contained some point not in B_i, (v, w) would contain two boundary points of $(\operatorname{conv} B_i) \sim B_i$, neither in (q'_i, q_{i+1}). Thus the line $L(v, w)$ would intersect both $[p, q'_i]$ and $[p, q_{i+1}]$, forcing $[v, w]$ to lie in B_i, a contradiction. Hence $[v, w] \subseteq B_i$.

In case v, w do not lie in the same A_i or B_i set, then since $\operatorname{conv}\{p, v, w\} \subseteq S$, no point of Q can lie interior to $\operatorname{conv}\{p, v, w\}$. Therefore $[v, w]$ must intersect each $[p, q_i]$ and $[p, q'_i]$ between $R(p, v)$ and $R(p, w)$, and $[v, w]$ can be written as a finite union of segments in S, each having end points in some A_i or B_i set. Therefore, by previous remarks, each of these segments is in T, and $[v, w] \subseteq T$. It is clear that Q_T is finite, since at most two lnc points are contributed by each $A_i \sim \{p\}$, and at most $m-2$ by each B_i.

Returning to a consideration of what will be needed to prove the theorem, in view of the Lawrence, Hare, Kenelly Theorem it suffices to prove that the set T just constructed is a union of $m-1$ convex sets. Since clearly $p \in \ker T$, $T \subseteq \operatorname{cl} H_1$, and Q_T is finite, it is therefore sufficient to prove the theorem under the assumption that S is bounded and Q is finite.

Lemma 2. *We may assume that p is not an lnc point for S.*

Proof of Lemma 2. By Lemma 1, we may assume that the set Q of lnc points of S is finite, so we may select some convex neighborhood N of p such that $N \cap Q \sim \{p\}$ is empty. Using standard arguments, it is easy to show that each component of $N \cap S \sim \{p\}$ has convex closure, and by remarks in [1], we may assume that no such component is a segment. Then using the m-convexity of S, clearly $S \sim \{p\}$ has at most $m-1$ components $S_1, \cdots, S_k$. Furthermore, it is easy to show that each set $\operatorname{cl} S_i$ has at most $m_i - 1$ visually independent points and is m_i-convex, where $2 \leqq m_i \leqq m$ and where $\Sigma_{i=1}^{k}(m_i - 1) = m-1$. Certainly $N \cap \operatorname{cl} S_i \sim \{p\}$ is convex, so p cannot be an lnc point for any $\operatorname{cl} S_i$. If we are able to show that each set $\operatorname{cl} S_i$ is decomposable into $m_i - 1$ convex sets, then S will be a union of $\Sigma_{i=1}^{k}(m_i - 1) = m - 1$ convex sets, finishing the argument. Hence it suffices to assume that $p \notin Q$, and the proof of Lemma 2 is complete.

Now repeat the construction used in the proof of Lemma 1 to define the collection $\mathcal{R}$ of rays. Since Q is finite, we may order the rays in a clockwise direction, letting W_i denote the closed subset of S determined by consecutive rays R_i and R_{i+1}, $1 \leqq i \leqq n$, where $R_1 \cup R_{n+1} = H$. By previous remarks, we may assume that $W_i = \mathrm{cl}(\mathrm{int}\, W_i)$, $1 \leqq i \leqq n$. Thus to each i, $2 \leqq i \leqq n-1$, there correspond two lnc points of S, denoted q_i, q'_i, where $q'_i = q_{i+1}$ for $1 \leqq i \leqq n-1$. It is easy to show that each W_i set is convex, and we call W_i a *wedge* of S.

Again by the Lawrence, Hare, Kenelly Theorem, we may assume that bdry W_i is polygonal and select segments $[q_i, p_i]$, $[p'_i, q'_i]$ in bdry W_i, $1 \leqq i \leqq n$ (where $q_1 \in R_1$, $q'_n \in R_{n+1}$ are selected in the manner indicated previously).

We decompose S by defining $\mathcal{U}_1, \cdots, \mathcal{U}_{m-1}$, each an appropriate collection of wedges of S. We assign wedges to the $\mathcal{U}_i$ sets in the following manner: Let W_1 be in $\mathcal{U}_1$, W_2 in $\mathcal{U}_2$, and let $\mathbf{P}_1 = \{\mathcal{U}_1\}$, $\mathbf{P}_2 = \{\mathcal{U}_1, \mathcal{U}_2\}$. Inductively, assume that each of the wedges $W_1, \cdots, W_j$ has been assigned to one of the sets $\mathcal{U}_1, \cdots, \mathcal{U}_l$, and that $\mathbf{P}_j = \{\mathcal{U}_1, \cdots, \mathcal{U}_l\}$ partitions these j wedges so that $\mathrm{conv}(\cup \mathcal{U}_i) \subseteq S$, $1 \leqq i \leqq l$. Let V_i denote the last wedge assigned to $\mathcal{U}_i$ (i.e., the wedge assigned to $\mathcal{U}_i$ having largest subscript). If necessary, relabel the V_i and corresponding $\mathcal{U}_i$ sets so that for $1 \leqq i_1 < i_2 \leqq l$, V_{i_1} precedes V_{i_2} in our ordering. We assign W_{j+1} in the following manner: If $\mathrm{conv}[W_{j+1} \cup (\cup \mathcal{U}_i)] \subseteq S$ for some i, choose i_0 to be the largest such subscript i, and assign W_{j+1} to $\mathcal{U}_{i_0}$. In this case, let $\mathbf{P}_{j+1} = \{\mathcal{U}_1, \cdots, \mathcal{U}_l\}$. If no such i exists, assign W_{j+1} to $\mathcal{U}_{l+1}$, and let $\mathbf{P}_{j+1} = \{\mathcal{U}_1, \cdots, \mathcal{U}_{l+1}\}$. In either case, $\mathbf{P}_{j+1}$ partitions the family $\{W_1, \cdots, W_{j+1}\}$. Since there are finitely many wedges, the inductive procedure must end in a finite number of steps, and we may assume that the last partition $\mathbf{P}_n = \{\mathcal{U}_1, \cdots, \mathcal{U}_k\}$ partitions the family $\{W_1, \cdots, W_n\}$ so that $\mathrm{conv}(\cup \mathcal{U}_i) \subseteq S$, $1 \leqq i \leqq k$. The integer k defined above will be called the *convex cover order* of S with respect to p. We will prove that $k = m - 1$.

For the remainder of the argument, we will let V_i denote the last wedge assigned to $\mathcal{U}_i$, $1 \leqq i \leqq k-1$, and let V_k denote the first wedge assigned to $\mathcal{U}_k$. Moreover, we relabel the $\mathcal{U}_i$ and corresponding V_i sets so that for $1 \leqq i_1 < i_2 \leqq k$, V_{i_1} precedes V_{i_2} in our clockwise ordering.

LEMMA 3. *In our assignment of wedges, for* $1 \leqq i < k$, $\mathrm{conv}(V_i \cup V_k) \not\subseteq S$.

PROOF OF LEMMA 3. Since $\mathrm{conv}(\cup \mathcal{U}_i) \subseteq S$, it is easy to see that

$\mathrm{conv}(V_k \cup (\cup \mathcal{U}_i))$ would lie in S, and in our partitioning procedure, V_k would have been assigned to $\mathcal{U}_j$ for some $1 \leqq j < k$, a contradiction.

LEMMA 4. *If S has convex cover order k with respect to p, then S contains k visually independent points $x_1, \cdots, x_k$ with $x_i \in (\mathrm{int}\, U_i \sim \mathrm{conv}(p, q_i, q'_i)) \cup (q_i, q'_i)$ for appropriate wedges $U_1, \cdots, U_k$ of S (ordered in a clockwise direction), $U_1 = V_1$, $U_k = V_k$. Moreover, x_1, x_k may be selected as close as we wish to q'_1, q_k respectively.*

PROOF OF LEMMA 4. Applying induction, the assertion is trivial for $k = 1, 2$, so we assume it true for all positive integers less than k, to prove for k. Consider the collections of wedges $\mathcal{U}_1, \cdots, \mathcal{U}_k$. By Lemma 3, $\mathrm{conv}(V_1 \cup V_k) \not\subseteq S$, and there are two cases arising from the different ways $\mathrm{conv}(V_1 \cup V_k)$ can contain points outside S: (1) For some wedge between V_1 and V_k, there is a corresponding member of Q which lies on the p side of $L(q'_1, q_k)$. (2) Case 1 does not occur and one of p'_1, p_k lies beyond $L(q'_1, q_k)$ from p.

Case 1. Suppose that for some wedge between V_1 and V_k, a corresponding member of Q lies on the p side of $L(q'_1, q_k) = L$. Among the q and q' points having this property, examine those whose distance to L is maximal, and from these select the one having largest subscript j. Assume it is $q'_{j-1} = q_j$ and corresponds to W_j.

Let M be the line through q_j parallel to L. Now the set S' defined as the union of V_1, W_{j-1}, and the wedges of S following V_1 and preceding W_{j-1} is clearly of the type considered in the hypothesis and has convex cover order $t < k$ according to the procedure involved in our assignment of wedges. (Clearly t is the largest integer l for which members of $\mathcal{U}_l$ are contained in S'.) Note that if $V_t \subseteq S'$, then V_t must be the last wedge in S', since every wedge following V_t has been assigned to $\mathcal{U}_j$ for some $j > t$ (by our labeling). By the induction hypothesis, S' contains points $x_1, \cdots, x_t$ visually independent via S' (and hence via S), and satisfying our specified requirements. Similarly let S'' be the union of V_k and the wedges of S following V_t and preceding V_k. Then S'' satisfies our hypothesis. Also, each of $V_{t+1}, \cdots, V_k$ lies in S'', so S'' has convex cover order at least $k - t$. Hence there are points $y_{t+1}, \cdots, y_k$ in S'' which are visually independent via S'' (and via S) and which satisfy our requirements. Certainly $\mathrm{int}(S' \cap S'') = \emptyset$, each of $x_1, \cdots, x_t$ precedes the ray $R(p, q_j)$, and each of $y_{t+1}, \cdots, y_k$ follows $R(p, q_j)$.

We assert that $x_1, \cdots, x_t, y_{t+1}, \cdots, y_k$ are visually independent via S: Clearly by our selection of q_j, for $1 < r \leqq t$, each x_r is in $S' \sim V_1$ and must lie on M or beyond M from p. For $t+1 \leqq s < k$, each y_s is in $S'' \sim V_k$ and must lie beyond M from p. Also, by our induction hypothesis, we may assume that x_1 and y_k are beyond M. Therefore, if $[x_r, y_s] \subseteq S$ for some $1 \leqq r \leqq t$ and some $t+1 \leqq s \leqq k$, then $q_j \in \operatorname{int}\operatorname{conv}\{p, x_r, y_s\} \subseteq \operatorname{int} S$, and q_j could not be an lnc point for S, clearly impossible. Hence the points are indeed visually independent. The remaining part of the inductive hypothesis is easy to verify, and the proof of Case 1 is complete.

Case 2. Suppose that Case 1 does not occur and that p_k lies beyond $L(q'_1, q_k) = L$ from p. (The proof for p'_1 beyond L is similar and will be omitted.) Let S' denote the union of V_1 and the wedges of S which follow V_1 and precede V_k. Then S' has convex cover order $k-1$ and we may apply our induction hypothesis to obtain visually independent points $x_1, \cdots, x_{k-1}$. Since every lnc point between q'_1 and q_k is on or beyond L, our inductive hypothesis assures us that $x_2, \cdots, x_{k-1}$ see no points on $[p_k, q_k)$ via S. Again by our hypothesis we may assume that x_1 sees no point on $[p_k, q_k)$ either. Then for an appropriate choice of x_k in V_k, $x_1, \cdots, x_k$ are visually independent via S. The rest of the argument is easy, finishing Case 2 and the proof of Lemma 4.

Now since S is m-convex, S has no more than m visually independent points. Thus by Lemma 4, S must have convex cover order $m-1$, and S is the union of the $m-1$ convex sets $S_i = \operatorname{conv}(\cup \mathcal{U}_i)$, $1 \leqq i \leqq m-1$, completing the proof of the theorem.

COROLLARY 1. *Let S be a closed set in the plane. Let $p \in \ker S \neq \emptyset$, and for H some line containing p, assume $S \subseteq \operatorname{cl} H_1$. Then for $m \geqq 2$, S is m-convex if and only if S is the union of $m-1$ convex sets.*

COROLLARY 2. *Let S be a closed set in the plane, Q the set of lnc points of S, with $p \in [(\operatorname{bdry} S) \cap (\ker S)] \sim Q \neq \emptyset$. Then for $m \geqq 2$, S is m-convex if and only if S is the union of $m-1$ convex sets.*

PROOF. Since $p \notin Q$, we may select a neighborhood N of p such that $N \cap S$ is convex. Then a hyperplane H supporting $N \cap S$ at p will have the required property.

COROLLARY 3. *Let S be a closed m-convex set in the plane with $p \in \ker S \neq \emptyset$. Then S is a union of $2(m-1) = 2m-2$ or fewer convex sets.*

PROOF. Let H be any line through p and apply Theorem 1 to $\operatorname{cl} H_1 \cap S$ and $\operatorname{cl} H_2 \cap S$.

3. Applications of Theorem 1

In this section we present three remarkably different kinds of decomposition theorems which may be proved from Theorem 1.

THEOREM 2. *Let S be a closed m-convex set in R^2 with $p \in \ker S \cap \operatorname{bdry} S \neq \emptyset$. Then S is the union of $2m-3$ or fewer convex sets.*

PROOF. The set S may obviously by expressed as the union of an m-convex set and an $(m-1)$-convex set, each satisfying the hypothesis of Theorem 1.

THEOREM 3. *If S is any closed 4-convex subset of the plane, then S is the union of 6 or fewer convex sets.*

PROOF. If S is not simply connected, then S is the union of 5 or fewer convex sets by a theorem of Guay [3]. Hence assume S is simply connected. Also, we may assume that S is connected, for otherwise the bound may be lowered to 4. In case $Q = \emptyset$, S is convex [7], and the result is trivial. For $Q \neq \emptyset$, select q in Q and define sets $S_q = \{x : [x, q] \subseteq S\}$ and $S^q = \{x : [x, q] \not\subseteq S\}$ (called the star and anti-star of q in S, respectively). The set S_q is closed and since $q \in \operatorname{bdry} S$, we have $q \in \ker S_q \cap \operatorname{bdry} S_q$. Using the simple connectedness of S, it is easy to show that S_q is 4-convex, and by Theorem 2, S_q is the union of $2 \cdot 4 - 3 = 5$ convex sets $C_1, \cdots, C_5$. Using the fact that $q \in Q$, it is easy to show that for x, y in S^q, $[x, y] \subseteq S$, and again by the simple connectedness of S, $C_6 = \operatorname{conv} S^q \subseteq S$. Hence $S = \bigcup_{i=1}^{6} C_i$, the desired result.

REMARK. By Example 3 in [5], the bound for a closed planar 4-convex set is no lower than 5. Hence the best bound is either 5 or 6.

The final two theorems of this section deal with the case in which $\operatorname{conv} Q \subseteq S$.

THEOREM 4. *Let S be a closed m-convex set in the plane, $m \geqq 2$, with Q the set of lnc points of S. If* $\operatorname{conv} Q \subseteq S$ *and* $\operatorname{int} \operatorname{conv} Q = \emptyset$, *then S is expressible as a union of $m-1$ convex sets. The bound is best possible for every m.*

PROOF. By comments in [1], we may assume that no component of $S \sim Q$ is a segment. Also, we may assume that Q is not a singleton set, for then the proof is easy. if S is 2-convex the result is trivial, and we assume the result is true when S is j-convex, $2 \leqq j < m$, to prove for m.

In case $S \sim Q$ is not connected, then $S \sim Q$ has at most $m-1$ components $S_1, \cdots, S_k$, $k \leqq m-1$. It is not hard to show that each set $\operatorname{cl} S_i$ has at most $m_i - 1$ visually independent points and is m_i-convex, where $2 \leqq m_i < m$ and where

$\Sigma_{i=1}^{k}(m_i - 1) = m - 1$. Then by our induction hypothesis each set $\mathrm{cl}\, S_i$ is a union of $m_i - 1$ convex sets, and S is a union of $\Sigma_{i=1}^{k}(m_i - 1) = m - 1$ convex sets, finishing the argument.

In case $S \sim Q$ is connected, then it is easy to show that S has $m - 2$ lnc points, and S is a union of $m - 1$ convex sets by a theorem of Guay and Kay [4, Theorem 1]. Clearly the bound of $m - 1$ is best possible, and the proof is complete.

THEOREM 5. *Let S be a closed m-convex set in the plane, $m \geqq 2$, with Q the set of lnc points of S. If* $\mathrm{conv}\, Q \subseteq S$, *then S is a union of $3m - 2$ or fewer convex sets.*

PROOF. The proof of this result is lengthy. First we shall show we may assume that Q is finite (Lemma 5), next that each component of $S \sim \mathrm{conv}\, Q$ is convex (Lemma 6), and that $S \sim Q$ is connected (Lemma 7). Then we define a subset T of S satisfying Theorem 1, Corollary 2, and show that the remaining points of S lie either in $\mathrm{conv}\, Q$ or in one of at most $2(m - 1)$ components of $S \sim \mathrm{conv}\, Q$. For the sake of brevity, some of the easy details of the argument are omitted.

Without loss of generality, assume S is connected, for every component of S not containing Q is necessarily convex. Also, it is clear that S must be simply connected. By earlier remarks, we may assume that S is bounded and that $S = \mathrm{cl}(\mathrm{int}\, S)$. And by Theorem 4, we may restrict our attention to the case in which $\mathrm{int}\,\mathrm{conv}\, Q \neq \emptyset$.

LEMMA 5. *To any finite subset $F = \{x_i : 1 \leqq i \leqq k\}$ of S there corresponds an m-convex set T having finitely many lnc points, with $F \subseteq T \subseteq S$. Hence we may assume that S has finitely many lnc points.*

PROOF OF LEMMA 5. Let $F = \{x_i : 1 \leqq i \leqq k\}$ be any finite subset of S, and without loss of generality, assume that the points of F are indexed so that $x_i \in S \sim \mathrm{conv}\, Q$ for $1 \leqq i \leqq n$ and $x_i \in \mathrm{conv}\, Q$ for $n \leqq i \leqq k$. By a lemma of Valentine [10, Lemma 1], each point in F sees some point of Q (and hence some point of $\mathrm{conv}\, Q$) via S. Moreover, if $x \notin \mathrm{conv}\, Q$, then x necessarily sees some point y in $\mathrm{bdry}\,\mathrm{conv}\, Q$ such that $[x, y) \cap \mathrm{conv}\, Q = \emptyset$. Therefore, for each i, $1 \leqq i \leqq n$, we may define $A_{x_i} = A_i = \{y : y$ in $\mathrm{bdry}\,\mathrm{conv}\, Q$, $[x_i, y] \subseteq S$ and $[x_i, y) \cap \mathrm{conv}\, Q = \emptyset\}$. Also, since S is simply connected and $\mathrm{conv}\, Q \subseteq S$, A_i is necessarily connected, so A_i is an arc in $\mathrm{bdry}\,\mathrm{conv}\, Q$.

We assert that the endpoints v_i, w_i (not necessarily distinct) of the arc A_i lie in Q: Let $a_i, b_i \in A_i$ and assume for the moment that a_i, b_i may be selected so that $a_i \neq b_i$. Then no point on A_i between a_i and b_i may lie in Q. In case $a_i \in Q$, then no point in $\mathrm{bdry}\,\mathrm{conv}\, Q$ beyond $L(x_i, a_i)$ from b_i may lie in A_i, and we may select

$v_i = a_i$. If $a_i \not\in Q$, then it is not hard to show that there are points of Q beyond $L(x_i, a_i)$ from b_i, and we may select such a point v so that the arc $\widehat{va_i}$ in bdry conv Q has minimal length. Then $[v, a_i] \cup [a_i, x_i] \subseteq S$, no point of Q is in $\text{conv}\{v, a_i, x_i\} \sim [v, x_i]$, so by a result of Valentine [8, Corollary 2], $\text{conv}\{v, a_i, x_i\} \subseteq S$, and x_i sees v via S. Then using the fact that $\widehat{va_i}$ has minimal length, it is easy to show that $v \in A_i$. Since $v \in Q$, x_i can see no point of bdry conv Q beyond $L(x_i, v)$ from a_i, and $v = v_i$ is the required point. A similar argument holds for b_i to produce w_i, finishing the argument. In case $\{a_i\} = \{b_i\} = A_i$, the previous argument may be adapted to show that $a_i \in Q$, and the assertion is proved.

For each i, $1 \leqq i \leqq n$, let W_i denote the component of $S \sim \text{conv}\, Q$ containing x_i, and let B_i denote the subset of conv Q corresponding to W_i — i.e., $B_i = \text{cl}\, W_i \cap \text{bdry conv}\, Q$. By earlier remarks, it is clear that for y in W_i, $A_y \subseteq B_i$. Moreover, since S is locally starshaped [5, Lemma 2], for s in B_i, there is some y in W_i such that $s \in A_y$, and $B_i = \cup\{A_y : y \text{ in } W_i\}$. Now for s, t in B_i, we may select y, z in W_i such that $[y, s] \cup [z, t] \subseteq S$. Since W_i is locally convex and connected, it is polygonally connected, and there is a path λ in $W_i \subseteq S \sim \text{conv}\, Q$ from s to t. Then λ cannot intersect $[s, t] \subseteq \text{conv}\, Q$, and it is easy to show that there is an arc in B_i from s to t. Clearly B_i is closed, so B_i is an arc $\widehat{q_iq'_i}$. Also, by earlier remarks, $q_i, q'_i \in Q$.

Define $T = (\bigcup_{i=1}^{n} W_i) \cup (\text{conv}\, Q)$. It is easy to show that T is an m-convex subset of S. Moreover, since S is locally starshaped and W_i is polygonally connected, an earlier argument may be adapted to show that the set of lnc points of T lies in $\{q_i, q'_i : \widehat{q_iq'_i} = \text{cl}\, W_i \cap \text{bdry conv}\, Q,\ 1 \leqq i \leqq n\}$. Therefore, by the Lawrence, Hare, Kenelly Theorem, it suffices to assume that Q is finite, finishing the proof of Lemma 5.

LEMMA 6. *Without loss of generality, we may assume that each component W_i of $S \sim$ conv Q is convex.*

PROOF OF LEMMA 6. Assume that some component W_i of $S \sim \text{conv}\, Q$ is not convex, and let $Q_i \neq \emptyset$ denote the set of lnc points of cl W_i. By the proof of Lemma 5, $Q_i \subseteq \{q_i, q'_i\}$, where $\widehat{q_iq'_i} = \text{cl}\, W_i \cap \text{bdry conv}\, Q$. Let L be a line which contains q_i, q'_i and which supports conv Q, and let L_1, L_2 denote the corresponding open halfspaces, with $\text{conv}\, Q \subseteq \text{cl}\, L_2$. Then using an argument employed in [4, lemma 6], $L_1 \cap W_i$ is convex and each of the two (or fewer) components of $L_2 \cap W_i$ is convex. If $L_2 \cap W_i$ has two components, then S may be written as the union of two convex sets and an $(m-2)$-convex set T. In case $L_2 \cap W_i$ is connected, then S is the union of a convex set and an $(m-1)$-convex set T. In either case, $T \cap W_i$ is convex, and without loss of generality we may assume that W_i is convex.

LEMMA 7. *Without loss of generality we may assume that $S \sim Q$ is connected.*

PROOF OF LEMMA 7. If $S \sim Q$ is not connected, then it is easy to show that S is expressible as the union of a convex set $\mathrm{cl}\, W_i$ and an $(m-1)$-convex set $\mathrm{cl}(S \sim W_i)$, for W_i a component of $S \sim Q$.

Returning to the proof of the theorem, order the lnc points of S and the corresponding components of $S \sim \mathrm{conv}\, Q$ in a clockwise direction along $\mathrm{bdry}\, \mathrm{conv}\, Q$. By Lemma 7, to each component W_i of $S \sim \mathrm{conv}\, Q$ there corresponds a pair q_i, q'_i of lnc points of S (where q'_i follows q_i in our ordering). By the Lawrence, Hare, Kenelly Theorem, we may assume that bdry W_i is polygonal, and hence we may select segments $[q_i, p_i]$,$[p'_i, q'_i]$ in bdry W_i. Let $L_i = L(q_i, p_i)$, $L'_i = L(q'_i, p'_i)$. We will say that a point x is *beneath* L_i if x is in the open halfspace L_{i1} determined by L_i and containing W_i. Similarly, x is *beyond* L_i if x is in the open halfspace L_{i2}.

Let W be any fixed component in $S \sim \mathrm{conv}\, Q$, and for convenience, assume $W = W_1$. We assert that $[q_1, q'_1]$ fails to be in $\mathrm{cl}(L_{i1})$ for at most $m-1$ of the lines L_i, $1 < i \leqq n$: Assume that $[q_1, q'_1] \not\subseteq \mathrm{cl}(L_{i1}) \cup \mathrm{cl}(L_{j1})$, where $1 < i < j \leqq n$. Then p_j sees no point of S beyond L_j. But q'_1 is necessarily beyond L_j, and hence q_i is, too (since if $q_i \neq q'_1$, then q_i follows q'_1 and precedes q_j in our ordering). Thus for c_i selected appropriately in (p_i, q_i), $[c_i, q_i]$ lies beyond L_j and no point of $[c_i, q_i)$ sees any point of $[p_j, q_j)$ via S. Then clearly for every collection of halfspaces $\mathrm{cl}(L_{i1})$ which fail to contain $[q_1, q'_1]$, there is a corresponding collection of visually independent points of S, so at most $m-1$ halfspaces have this property. Let $\mathscr{A}$ denote the associated collection of components of $S \sim \mathrm{conv}\, Q$.

Similarly, letting $\mathscr{B}$ denote the collection of components W_i of $S \sim \mathrm{conv}\, Q$ for which $[q_1, q'_1] \not\subseteq \mathrm{cl}(L'_{i1})$, then $\mathscr{B}$ has at most $m-1$ members. Define $T = \mathrm{cl}(S \sim \cup\{W\colon W = W_1,\ W \in \mathscr{A} \text{ or } W \in \mathscr{B}\})$. For $W \notin \mathscr{A} \cup \mathscr{B}$, every point of W sees $[q_1, q'_1]$ via T, and for t in (q_1, q'_1), t is $\mathrm{bdry}\, T \cap \ker T$. Since T is a closed m-convex set, we may use Theorem 1, Corollary 2, to conclude that T is a union of $m-1$ convex sets. Therefore, S is a union of $3(m-1)+1 = 3m-2$ or fewer convex sets, finishing the proof of Theorem 5.

4. The general case

A general decomposition theorem will require several preliminary lemmas.

LEMMA 8. *Let S be a closed m-convex set in the plane. If B is the closure of a* *bounded component A of $R^2 \sim S$, then* conv B *is a polygon having at most $m-1$ sides.*

Proof of Lemma 8. Certainly conv B is the convex hull of its extreme points. To see that conv B is a polygon, we show that it has at most $2m-1$ (and hence finitely many) extreme points: If conv B had $2m$ extreme points, they could be ordered in a clockwise direction along bdry conv B. Letting $x_1, x_2, \cdots, x_{2m}$ denote these points, clearly the set $\{x_{2k} : 1 \leqq k \leqq m\}$ would be a set of m visually independent points of S, for otherwise A could not be connected. However, this would contradict the m-convexity of S. Thus conv B may have at most $2m-1$ extreme points.

It remains to show that the polygon conv B has at most $m-1$ sides. Let $x_1, x_2, \cdots, x_k$ denote the vertices of conv B, $k \geqq 3$, where the points are again ordered in a clockwise direction along bdry conv B. Then $A \subseteq B \subseteq$ conv B. We will select k visually independent points $y_1, \cdots, y_k$ of S. (For convenience of notation, let $x_{k+1} = x_1$.) If (x_i, x_{i+1}) contains a point in S, let $y_i \in (x_i, x_{i+1}) \cap S$. Otherwise, (x_i, x_{i+1}) lies in a (possibly unbounded) component of $R^2 \sim S$, and this component is distinct from A since $A \subseteq$ conv B. Hence it is not hard to see that there is some component S_i of $S \sim \{x_i, x_{i+1}\}$ which lies in conv B such that $x_i, x_{i+1} \in$ cl S_i. In this case, select $y_i \in S_i$. Then $y_1, \cdots, y_k$ is a visually independent subset of S, for otherwise A could not be connected. Therefore $k \leqq m-1$ and the proof of Lemma 8 is complete.

Lemma 9. *Let $S =$ cl int S be a set in the plane. If $R^2 \sim S$ has at least $r = (n+2)2^{n-1}$ bounded components having closures $B_1, \cdots, B_r$, $n \geqq 0$, and for each i,* conv B_i *is a convex polygon, then S has at least $n+3$ visually independent points on $\bigcup_{i=1}^r$* bdry B_i.

Proof of Lemma 9. The proof is by induction. If $n = 0$, then $r = 1$and clearly S has at least three visually independent points. Assume the result is true for integers less than n, $n \geqq 1$, to prove for n.

Consider the polygon $P \equiv$ conv $(\bigcup_{i=1}^r B_i)$ and let p be any extreme point of P. Then p is an extreme point of some conv B_i, say of conv B_r. Choose a line H supporting P such that $H \cap P = \{p\}$. Choose a line L through p which intersects int B_r. Now if $n+2$ of the sets $B_1, \cdots, B_r$ share a segment with L, then we can find $n+3$ visually independent points in $\bigcup_{i=1}^r$ bdry B_i, finishing the argument. For since $S =$ cl int S, no two B sets share a segment. To each B_i sharing a segment with L, we may associate points p_i, p_i' on L with $B_i \cap L$ containing $[p_i, p_i']$ and p_i, p_i' in bdry B_i. Also, we may relabel the B sets and corresponding p

points so that $p_1 < p_1' \leqq p_2 < \cdots \leqq p_{n+2} < p'_{n+2}$. Clearly by selecting points x_i in bdry $B_i \sim L$, x_i sufficiently close to p_i, and y_{n+2} in bdry $B_{n+2} \sim L$, y_{n+2} sufficiently close to p'_{n+2}, we have $x_1, \cdots, x_{n+2}, y_{n+2}$ a set of $n+3$ visually independent points of S.

Hence we may assume that L meets at most $n+1$ of the sets $B_1, \cdots, B_r$ in a segment. Then from these r sets there are at least

$$r-(n+1)=(n+2)2^{n-1}-n-1$$

not sharing a segment with L, and each of these sets must lie entirely in one of the closed halfspaces determined by L. Hence one of these halfspaces, say $\mathrm{cl}(L_1)$, must contain at least $(r-n-1)/2$ of the B_i sets, $1 \leqq i \leqq r-n-1$.

Consider the set

$$S' = S \cup (\cup \{B_i : 1 \leqq i \leqq r, B_i \not\subseteq \mathrm{cl}(L_1)\}).$$

Then S' is a closed set having at least $r' \geqq (r-n-1)/2$ bounded components which satisfy the hypothesis of the theorem. Moreover

$$\frac{r-n-1}{2} = \frac{(n+2)2^{n-1}-n-1}{2}$$

$$= (n+1)2^{n-2} + 2^{n-2} - \frac{n}{2} - \frac{1}{2}.$$

If $n \geqq 3$, then $2^{n-2} \geqq n/2 + 1/2$, and thus $r' \geqq (r-n-1)/2 \geqq (n+1)2^{n-2}$. In case $n=1$, then $(r-n-1)/2 = (3-2)/2 = 1/2$, and since r' is an integer, $r' \geqq 1 = (n+1)2^{n-2}$. Similarly, if $n=2$ then $(r-n-1)/2 = (4\cdot 2-3)/2 = 5/2$, and $r' \geqq 3 = (n+1)2^{n-2}$. We conclude that for $n \geqq 1$, S' has at least $(n+1)2^{n-2}$ bounded components in its complement. Therefore, by the induction hypothesis, S' has at least $n+2$ visually independent points $x_1, \cdots, x_{n+2}$ on $\cup\{\mathrm{bdry}\, B_i : 1 \leqq i \leqq r\} \cap \mathrm{cl}(L_1)$, and these points are also visually independent via S.

We assert that we may choose a point x_{n+3} on bdry B_r which sees none of the points $x_1, \cdots, x_{n+2}$ via S: Recall that the line L contains points interior to B_r. Let q denote the vertex of the polygon conv B_r which lies in the open halfspace L_2 and for which $[p,q] \subseteq \mathrm{bdry\,conv}\, B_r$. In case $(p,q) \cap \mathrm{bdry}\, B_r \neq \varnothing$, then let x_{n+3} be any member of this set. Otherwise, fix $y \in (p,q)$ and consider the set $\{x : x \in \mathrm{bdry}\, B_r \text{ and } (y,x) \cap B_r \neq \varnothing\}$. Then let x_{n+3} be any member of this set in L_2 and distinct from q. Clearly x_{n+3} sees no point of S in $\mathrm{cl}(L_1)$, and $x_1, \cdots, x_{n+3}$ is a set of visually independent points. This completes the induction and finishes the proof of Lemma 9.

COROLLARY 1. *If $S = \operatorname{cl}\operatorname{int} S$ is an m-convex set in the plane, then $R^2 \sim S$ has no more than $(m-1)2^{m-4}-1$ bounded components.*

PROOF. By Lemma 8, if B is the closure of a bounded component of $R^2 \sim S$, then $\operatorname{conv} B$ is a polygon (having at most $m-1 \geqq 3$ edges). Then by Lemma 9, if $R^2 \sim S$ has r bounded components, assuming $(n+2)2^{n-1} \leqq r < (n+3)2^n$, $n \geqq 0$, then S has at least $n+3$ visually independent points. Since $n+3 \leqq m-1$, we have $r < (m-1)2^{m-4}$, the desired result.

LEMMA 10. *Let $S = \operatorname{cl}\operatorname{int} S$ be an m-convex set in the plane, $m \geqq 3$, with $x \in S$. Then the set $S_x = \{y : [x,y] \subseteq S\}$ is k-convex, where $2 \leqq k \leqq (m-1)^2 2^{m-4}+1$.*

PROOF OF LEMMA 10. Suppose on the contrary that S_x contains at least $k = (m-1)^2 2^{m-4}+1 \geqq m$ visually independent points $x_1, \cdots, x_k$. Let $A_1, \cdots, A_r$ denote the bounded components of $R^2 \sim S$, where $0 \leqq r \leqq (m-1)2^{m-4}-1$ by the corollary to Lemma 9. For each nonempty set A_i, select a point b_i in A_i and examine the rays $R(x, b_i)$. Order the rays in a clockwise direction and relabel the A_i sets so that $R(x, b_{i+1})$ follows $R(x, b_i)$ in our ordering.

The rays define closed subset of the plane: If $r \geqq 2$, let T_i be the closed subset determined by $R(x, b_i)$ and $R(x, b_{i+1})$ relative to our clockwise orientation, $1 \leqq i \leqq r$ (where $r+1 \equiv 1$). At most one T_i set is not convex, and if this occurs, bisect the corresponding angle to yield two convex sets, T_{i1} and T_{i2}. Hence we obtain either r or $r+1$ closed convex T sets. In case $r=1$, let T_1, T_2 be the closed halfspaces determined by the line $L(x, b_1)$, and if $r=0$, let T_1 be the plane. Clearly in all cases $T_i \cap S$ is simply connected for each i. (Otherwise some ray $R(x, b_j)$ would lie between $R(x, b_i)$ and $R(x, b_{i+1})$ in our ordering, impossible.)

We assert that at least m of the points $x_1, \cdots, x_k$ lie in one of the convex T regions: If fewer than m of the points $x_1, \cdots, x_k$ belonged to T_i for each i, then there would be at most $(m-1)(r+1) \leqq (m-1)^2 2^{m-4} < k$ points in all, a contradiction.

Hence one of the regions, say T_1, contains m of the x points. For convenience, say $x_1, \cdots, x_m \in T_1$. By m-convexity of S, at least one corresponding segment, say $[x_1, x_2]$, is in S. Hence $[x, x_1] \cup [x, x_2] \cup [x_1, x_2] \subseteq S$ with $[x_1, x_2] \not\subseteq S_x$, denying simple-connectedness of $T_1 \cap S$. Thus S_x is indeed k-convex, $2 \leqq k \leqq (m-1)^2 2^{m-4}+1$, finishing the proof of Lemma 10.

THEOREM 6. *If S is any closed m-convex set in the plane, $m \geqq 3$, then S is the union of $(m-1)^3 2^{m-3}$ or fewer convex sets.*

PROOF. Without loss of generality, we may assume that $S = \mathrm{cl\,int}\, S$, for otherwise S is the union of k segments and an $(m-k)$-convex set for some $1 \leqq k \leqq m-2$.

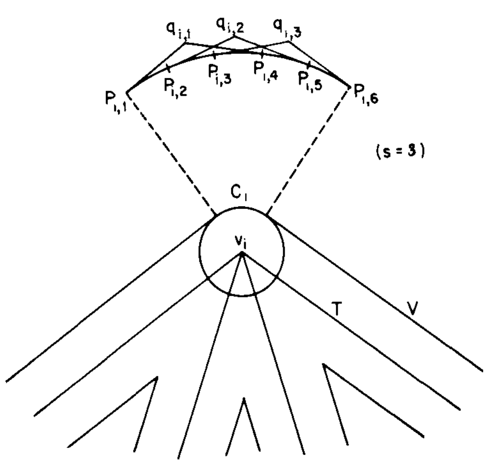
By [5, Theorem 2], there exist $m-1$ or fewer points $x_1, \cdots, x_{m-1}$ in S such that $S = \bigcup_{i=1}^{m-1} S_i$, where $S_i = \{y : [x_i, y] \subseteq S\}$. By Lemma 10, each S_i is at most $[(m-1)^2 2^{m-4} + 1]$-convex. Since $x_i \in \ker S_i \neq \emptyset$, by Corollary 3 to Theorem 1, each S_i is a union of $2[(m-1)^2 2^{m-4}]$ or fewer convex sets. Thus S is the union of $(m-1)[2(m-1)^2 2^{m-4}] = (m-1)^3 2^{m-3}$ or fewer convex sets.

COROLLARY 1. *A closed set S in the plane is m-convex for some $m \geqq 2$ if and only if S is the union of finitely many closed convex sets.*

5. An example

M. A. Perles has communicated the following example of a class of compact m-convex subsets of E^2, mentioned in an earlier paper of one of the authors [5].

Example. For each integer $r \geqq 2$ and $s \geqq 1$ a set $S_{r,s}$ will be defined by first taking the vertices $v_1, \cdots, v_r$ of a regular polygon in E^2 inscribed in a unit circle, and setting $T = \cup\{[v_i, v_j] \mid 1 \leqq i < j \leqq r\}$. With $0 < \delta < \pi/10r$ put $V = T + \delta B$, where B is the unit disk. Hence, V consists of $\binom{r}{2}$ parallel strips of width 2δ joining one another at the points v_i, with the outer corners being rounded off by disks of radius δ centered at the v_i, $i = 1, \cdots, r$. If K is the boundary of the set $\mathrm{conv}\, V$ then K consists of segments parallel to $[v_i, v_{i+1}]$ and circular arcs C_i of radius δ, $i = 1, \cdots, r$, where each C_i is less than a semicircle. Now divide each C_i into $2s-1$ equal sub arcs with consecutive points of division labeled $p_{i,1}, \cdots, p_{i,2s}$ (see figure). For each pair $(p_{i,j}, p_{i,s+j})$ let the tangents to C_i at $p_{i,j}$ and $p_{i,s+j}$ meet at



$q_{i,j}$, $j = 1, \cdots, s$, and put $\Delta_{i,j} = \operatorname{conv}\{p_{i,j}, p_{i,s+j}, q_{i,j}\}$. Finally, define $S_{r,s} = V \cup (\cup\{\Delta_{i,j} \mid 1 \leqq i \leqq r, 1 \leqq j \leqq s\}$.

It can be easily verified that $S_{r,s}$ has the following properties, for each $r \geqq 2$, $s \geqq 1$:

1) The points $q_{i,1}, \cdots, q_{i,s}$ are visually independent via $S_{r,s}$.

2) One may associate a point with each of the $\binom{r}{2}$ parallel strips so that the resulting $\binom{r}{2}$ points are visually independent via $S_{r,s}$.

3) Starting with the s points $q_{i,1}, \cdots, q_{i,s}$ for any i, points may be associated with each of the remaining $\binom{r-1}{2}$ parallel strips not passing through v_i yielding $s + \binom{r-1}{2}$ visually independent points.

4) $S_{r,s}$ is m-convex, where $m = 1 + \max\left\{\binom{r}{2}, s + \binom{r-1}{2}\right\}$.

5) Each of the points $q_{i,k}$ can see each of $q_{j,l}$ via $S_{r,s}$ for each $i \neq j$, $1 \leqq k \leqq s$, $1 \leqq l \leqq s$; consequently, $\operatorname{conv}(\Delta_{i,k} \cup \Delta_{j,l}) \subset S_{r,s}$ for $1 \leqq i \leqq r$, $1 \leqq j \leqq r$, $1 \leqq k \leqq s$, $1 \leqq l \leqq s$, and $k \neq l$.

6) If r is even and $s \geqq r$, in order to cover $S_{r,s}$ with the least number of convex subsets, choose the $\binom{r}{2}$ parallel strips together with one $\Delta_{i,j}$ at each end per strip, leaving $s - r + 1$ sets $\Delta_{i,j}$ not accounted for at each v_i. Opposite pairs of these remaining $\Delta_{i,j}$ can be taken into convex subsets inside $r/2$ parallel strips yielding

$$\binom{r}{2} + (s - r + 1)\frac{r}{2} = \frac{rs}{2} = \left[\frac{rs+1}{2}\right]$$

convex sets. A similar analysis yields the same number when r is odd.

7) If $s < r$ then all the sets $\Delta_{i,j}$ can be included with the $\binom{r}{2}$ parallel strips.

8) Since $\left[\frac{rs+1}{2}\right] \leqq \binom{r}{2}$ if and only if $s \leqq r - 1$, $S_{r,s}$ is the union of n closed, convex sets and is not the union of fewer than n, where

$$n = \max\left\{\binom{r}{2}, \left[\frac{rs+1}{2}\right]\right\}.$$

Note that if $s < r$ then $S_{r,s}$ is an example of an m-convex set which is the union of $m - 1$ but no fewer convex sets, since in this case $m - 1 = \binom{r}{2} = n$. But consider the set $S = S_{r,s}$ where $s = r^2 \geqq r$ $(r \geqq 2)$. S is then a compact, planar m-convex set with

$$m = 1 + s + \binom{r-1}{2} = \frac{3r^2 - 3r + 4}{2}, \quad n = \left[\frac{rs+1}{2}\right] \geqq \frac{r^3}{2}.$$

It then follows that

$$\sqrt{\frac{2}{27}} m^{3/2} < n < \sqrt{\frac{2}{9}} m^{3/2},$$

so S is the union of less than $m^{3/2}$ convex sets but is not the union of $(1/4)m^{3/2}$ convex sets, and m can assume the values of the infinite sequence $5, 11, 20, 32, \cdots$.

Hence, the best possible bound, while possibly not as large as $(m-1)^3 2^{m-3}$, cannot be linear in m, in general. In higher dimensions, the situation is infinitely worse since Perles has also constructed an example of a compact 3-convex subset of E^4 which is not the union of a finite number of convex sets.

References

1. Marilyn Breen, *A decomposition theorem for m-convex sets*, Israel J. Math. **24** (1976), 211–216.

2. H. G. Eggleston, *A condition for a compact plane set to be a union of finitely many convex sets*, Proc. Cambridge Phil. Soc. **76** (1974), 61–66.

3. M. D. Guay, *Planar sets having property P_m*, Doctoral Dissertation, Michigan State University, East Lansing, 1967.

4. Merle D. Guay and David C. Kay, *On sets having finitely many points of local nonconvexity and property P_m*, Israel J. Math. **10** (1971). 196–209.

5. David C. Kay and Merle D. Guay, *Convexity and a certain property P_m*, Israel J. Math. **8** (1970), 39–52.

6. J. F. Lawrence, W. R. Hare and John W. Kenelly, *Finite unions of convex sets*, Proc. Amer. Math. Soc. **34** (1972), 225–228.

7. W. L. Stamey and J. M. Marr, *Unions of two convex sets*, Canad. J. Math. **15** (1963), 152–156.

8. H. Tietze, *Über Konvexheit im kleinen und im grossen und über gewisse den Punkten einer Menge zugeordnete Dimensionzahlen*, Math. Z. **28** (1928), 697–707.

9. F. A. Valentine, *A three point convexity property*, Pacific J. Math. **7** (1957), 1227–1235.

10. F. A. Valentine, *Local convexity and L_n sets*, Proc. Amer. Math. Soc. **16** (1965), 1305–1310.

Department of Mathematics
University of Oklahoma
Norman Oklahoma 73019, USA